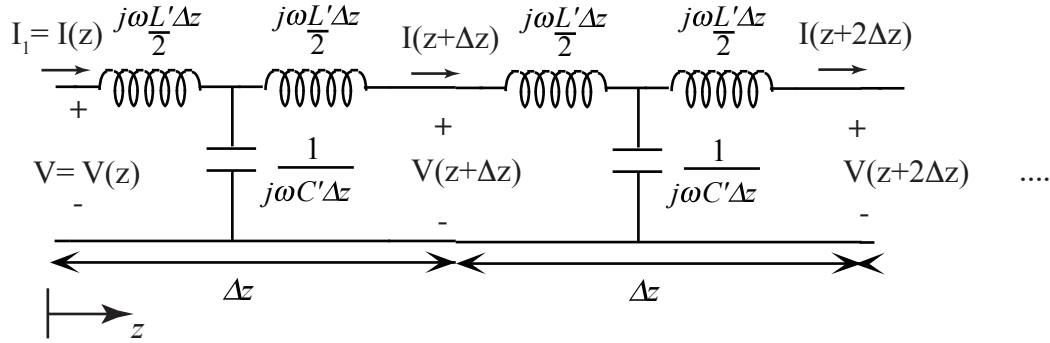
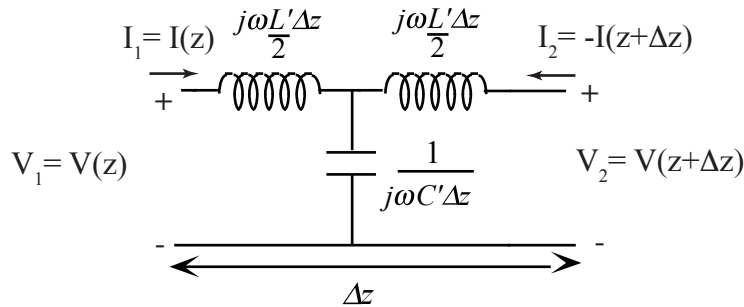


## Transmission Line Analysis using Transmission Parameters

Copnsidering a uniform transmission line to be a continuous chain of T-cell equivalent circuits:



Here,  $L'$  and  $C'$  are the inductance per meter and capacitance per meter of the transmission line, respectively, and both the voltage and current change from cell to cell. In the limit as  $\Delta z \rightarrow 0$ , these become continuous functions  $V(z)$  and  $I(z)$ , respectively. Considering a single cell as a 2-port network, we have:



Since this is a T-circuit, the Z parameters are easy to obtain:

$$\mathbf{Z} = \begin{bmatrix} \left( j\omega \frac{L'}{2} \Delta z + \frac{1}{j\omega C' \Delta z} \right) & \frac{1}{j\omega C' \Delta z} \\ \frac{1}{j\omega C' \Delta z} & \left( j\omega \frac{L'}{2} \Delta z + \frac{1}{j\omega C' \Delta z} \right) \end{bmatrix}$$

From these Z parameters, the transmission (ABCD) parameters can be found to be:

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} = \begin{bmatrix} \left(1 - \omega^2 \frac{L'}{2} C' \Delta z^2\right) & \frac{j\omega L' \Delta z}{2} \left(2 - \omega^2 \frac{L'}{2} C' \Delta z^2\right) \\ j\omega C' \Delta z & \left(1 - \omega^2 \frac{L'}{2} C' \Delta z^2\right) \end{bmatrix}$$

so

$$\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{pmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{pmatrix}, \text{ which can be written as:}$$

$$\begin{pmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{pmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{V}(z + \Delta z) \\ \mathbf{I}(z + \Delta z) \end{pmatrix}$$

We now ask the question: are there a “natural” voltage/current modes where the ratios of the voltages and currents (in the same direction) are the same at the input and output of the cells?

This would mean:

$$\begin{pmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{pmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{V}(z + \Delta z) \\ \mathbf{I}(z + \Delta z) \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{V}(z + \Delta z) \\ \mathbf{I}(z + \Delta z) \end{pmatrix}$$

Which means that we are looking for a combination  $\begin{pmatrix} \mathbf{V}(z) \\ \mathbf{I}(z) \end{pmatrix}$  that is an eigenvector of the T-matrix. This requires:

$$\begin{vmatrix} (\mathbf{T}_{11} - \lambda) & \mathbf{T}_{12} \\ \mathbf{T}_{21} & (\mathbf{T}_{22} - \lambda) \end{vmatrix} = 0 \quad ,$$

or

$$\lambda^2 - \lambda(\mathbf{T}_{11} + \mathbf{T}_{22}) + (\mathbf{T}_{11}\mathbf{T}_{22} - \mathbf{T}_{12}\mathbf{T}_{21}) = 0$$

Substituting the values of the T-parameters, this becomes the eigen equation:

$$\lambda^2 - 2\left(1 - \omega^2 \frac{L'}{2} C' \Delta z^2\right)\lambda + 1 = 0 \quad .$$

As long as  $\Delta z$  is small,

$$1 - \omega^2 \frac{L'}{2} C' \Delta z^2 \rightarrow \cos\left(\omega \sqrt{L' C'} \Delta z\right) = \cos\left(\omega \frac{\Delta z}{u}\right)$$

Where  $u = \frac{1}{\sqrt{L'C'}}$  is a *velocity*. So, the eigen equation for small  $\Delta z$  becomes

$$\lambda^2 - 2 \cdot \lambda \cdot \cos\left(\omega \frac{\Delta z}{u}\right) + 1 = 0$$

Using the quadratic equation, the resulting eigenvalues are;

$$\lambda = \cos\left(\omega \cdot \frac{\Delta z}{u}\right) \pm j \cdot \sin\left(\omega \cdot \frac{\Delta z}{u}\right) = e^{\pm j\omega \cdot \frac{\Delta z}{u}}$$

Using these eigenvalues, we can find the eigenvectors from:

$$T_{11}\mathbf{V} - T_{12}\mathbf{I} = \lambda\mathbf{V}$$

$$T_{21}\mathbf{V} - T_{22}\mathbf{I} = \lambda\mathbf{I}$$

From the second equation, we have:

$$j\omega C' \Delta z \cdot \mathbf{V} + \left(1 - \omega^2 \frac{L'}{2} C' \Delta z^2\right) \mathbf{I} = \mathbf{I} \cdot e^{\pm j\omega \frac{\Delta z}{u}} \approx \mathbf{I} \cdot \left[\left(1 - \omega^2 \frac{L'}{2} C' \Delta z^2\right) \pm j\omega \sqrt{L'C'}\right]$$

$$\text{which becomes: } \mathbf{V} = \pm \sqrt{\frac{L'}{C'}} \mathbf{I} = \pm Z_0 \mathbf{I}$$

where  $Z_0$  is called the characteristic impedance of the line. Hence, the eigenvectors are:

$$\begin{pmatrix} \mathbf{V}_+ \\ \mathbf{I}_+ \end{pmatrix} = \begin{pmatrix} \mathbf{V} \\ \mathbf{V}/Z_0 \end{pmatrix} \text{ for } \lambda = e^{j\omega \frac{\Delta z}{u}}$$

$$\begin{pmatrix} \mathbf{V}_- \\ \mathbf{I}_- \end{pmatrix} = \begin{pmatrix} \mathbf{V} \\ -\mathbf{V}/Z_0 \end{pmatrix} \text{ for } \lambda = e^{-j\omega \frac{\Delta z}{u}}$$

To see the behavior of these eigen vectors (modes), we note that:

$$\begin{pmatrix} \mathbf{V}_+(z) \\ \mathbf{I}_+(z) \end{pmatrix} = e^{s \frac{\Delta z}{u}} \cdot \begin{pmatrix} \mathbf{V}_+(z + \Delta z) \\ \frac{1}{Z_0} \mathbf{V}_+(z + \Delta z) \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{V}_-(z) \\ \mathbf{I}_-(z) \end{pmatrix} = e^{-s \frac{\Delta z}{u}} \cdot \begin{pmatrix} \mathbf{V}_-(z + \Delta z) \\ \frac{-1}{Z_0} \mathbf{V}_-(z + \Delta z) \end{pmatrix}$$

Where we have replaced  $j\omega$  with the complex frequency  $s$ . Remembering the Laplace time-shift relation  $f(t \pm t_0) \Leftrightarrow e^{\pm st_0} \cdot F(s)$ , where  $f(t)$  and  $F(s)$  are Laplace transform pairs, we have

$$\begin{pmatrix} v_+(t, z) \\ i_+(t, z) \end{pmatrix} = \begin{pmatrix} v_+(t + \Delta t, z + \Delta z) \\ \frac{1}{Z_0} v_+(t + \Delta t, z + \Delta z) \end{pmatrix}$$

and

$$\begin{pmatrix} v_-(t, z) \\ i_-(t, z) \end{pmatrix} = \begin{pmatrix} v_-(t - \Delta t, z + \Delta z) \\ \frac{-1}{Z_0} v_-(t - \Delta t, z + \Delta z) \end{pmatrix}$$

Where  $\Delta t = \frac{\Delta z}{u}$ , which means that the values of the  $v_+$  and  $i_+$  at the input (i.e.  $z$ ) of each T-cell at time  $t$  appear delayed in time at the output (i.e.,  $z + \Delta z$ ). Alternately, the values of the  $v_-$  and  $i_-$  at the input (i.e.  $z$ ) of each T-cell at time  $t$  appear advanced in time at the output (i.e.,  $z + \Delta z$ ). And, since each cell is identical, the total delay or advance for the eigen modes over a length  $z$  will be  $z/u$ . Hence, the eigenmodes behave as:

$$\begin{pmatrix} v_+(t, z) \\ i_+(t, z) \end{pmatrix} = \begin{pmatrix} v_+\left(t - \frac{z}{u}\right) \\ \frac{1}{Z_0} v_+\left(t - \frac{z}{u}\right) \end{pmatrix}$$

and

$$\begin{pmatrix} v_-(t, z) \\ i_-(t, z) \end{pmatrix} = \begin{pmatrix} v_-\left(t + \frac{z}{u}\right) \\ \frac{-1}{Z_0} v_-\left(t + \frac{z}{u}\right) \end{pmatrix}$$

Where the functions  $V_+$  and  $V_-$  are actually functions of single variables  $t - \frac{z}{u}$  and  $t + \frac{z}{u}$ , respectively.